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Abstract: This paper addresses a method for robust fault detection and estimation by minimizing the disturbance and uncertainties to residual sensitivity. It consists in the design of proportional integral observer while minimizing the well known $\mathcal{H}_\infty$ norm for worst case uncertainties and disturbance attenuation, and combining a transient response specification. This multi-objective problem is formulated as linear matrix inequalities (LMI) feasibility problem in which a cost function is minimized subject to LMI constraints. This approach is employed to generate a set of robust observers for uncertain switched systems.

Keywords: PI Observer, robust fault detection and estimation, uncertain switched systems, $\mathcal{H}_\infty$, LMI, MLF.

1. INTRODUCTION

Modern systems (vehicles, aircrafts, trains...) are increasingly equipped with new mechanisms to improve passengers safety Isermann (2005); Venkatasubramanian et al. (2003); Koenig et al. (2013). These new systems have often active parts using data from sensors and actuator, for detection and diagnosis of process faults.

There is an abundance of literature on fault detection (FD) techniquesChen and Patton (1999); Varrier et al. (2014). The idea is to compute a residual signal that represents the inconsistency between the actual plant variables and the mathematical model, to extract information on possible changes caused by faults Zhang (2009). In practical applications, the residuals are corrupted by unknown inputs such as noises, disturbances, and uncertainties in the system model. Hence, the main objective of FD methods is to generate stable robust residuals that are insensitive to these noise and uncertainties, while sensitive to faultsShi and Patton (2012); Liu and Zhou (2007).

The Linear Matrix Inequalities (LMI) formulation is often used to mathematically express robust fault detection problem, for classes of uncertain systems with bounded uncertainties, or non-linear system with Lipschitz non-linearities. The idea of finding a formulation with bounded unknowns has been widely studied. Some authors have proposed adaptive observerPaesa et al. (2010); Pourgholi and Majd (2013), adding an adaptive term to the observer. Nevertheless, there are two potential problems in their implementation: first the adaptive term might increase unboundedly and become infeasible in computationGu and Yang (2011). Second, it is an online adaption, that leads to more calculation and power consumption. Therefore, instead of using an adaptive term in the observer, the proposed method is to extend the Lyapunov function into a Multiple Lyapunov Function (MLF) resulting to a feasible LMI problem.

The specifications and objectives under consideration include $\mathcal{H}_\infty$ performance and time domain constraints. The motivations for using this mixed performances are as follows:

- The time domain constraint that is expressed by pole region assignment is useful to tune the transient response Moore (1976); Patton and Chen (1991, 1997); Liu and Patton (1998).
- The integral term in the fault observer is convenient to ensure a zero fault estimation error in steady state regime.
- The $\mathcal{H}_\infty$ performance is useful to ensure the residual robustness to model uncertainties, disturbances and unknown inputs.

In this paper, a proportional integral observer based filter is designed with the mixed $\mathcal{H}_\infty$ / eigen region assignment objectives. The desired observer is computed by solving a set of LMIs. A compromise between fault sensitivity, unknown input rejection, uncertainty robustness and eigen region assignment is optimized via a convex optimization algorithm.

The outline of this paper is as follows. After the introduction, problem formulation is given in Section II. In section III, preliminaries for the $\mathcal{H}_\infty$ synthesis, the eigen region synthesis and the fault Proportional integral (PI) observer formulation. The multi-objective switched robust fault detection observer scheme is given in Section IV. The set of LMIs are then solved as an optimization problem.

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The above results are illustrated by a numerical example in Section V with filed data from a car as application to lateral vehicle control. Finally, Section VI shows the concluding remarks and the possible future work.

Notations: The notation used in this paper is standard. X^T is the transposed of matrix X, the star symbol $(\star)$ in a symmetric matrix denotes the transposed block in the symmetric position. The notation $P > (<) 0$ means P is real symmetric positive (negative) definite matrix. 0 and I denote zeros and identity matrix of appropriate dimensions.

2. PROBLEM FORMULATION

Consider the state space representation of the linear time uncertain switched system :

$$\begin{cases} \dot{x}(t) = \bar{A}_{\alpha(t)}x(t) + B_{\alpha(t)}u(t) \\ \qquad +E_{d,\alpha(t)}d(t) + E_{f,\alpha(t)}f(t) \\ y(t) = C_{\alpha(t)}x(t) + D_{\alpha(t)}u(t) \\ \qquad +F_{d,\alpha(t)}d(t) + F_{f,\alpha(t)}f(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measurement output vector, $u \in \mathbb{R}^m$ is the input vector, $d \in \mathbb{R}^{n_d}$ is the disturbance vector, $f \in \mathbb{R}^{n_f}$ is the vector of faults to be detected, $\alpha(t)$ is the switching signal, it is assumed known and measured.

Model uncertainties can be represented in different forms, in this study additive form is considered as:

$$\bar{A}_{\alpha(t)} = A_{\alpha(t)} + \Delta_{x,\alpha(t)}N_{x,\alpha(t)} \tag{2}$$

The matrices A_α, B_α, $E_{d,\alpha}$, $E_{f,\alpha}$, C_α, D_α, $F_{d,\alpha}$ and $F_{f,\alpha}$ are the nominal LTI system matrices, they are known and in appropriate dimensions. $\Delta_{x,\alpha}$ is the state uncertainty matrix that is bounded $\|\Delta_{x,\alpha}\|_2 \leq \epsilon_{x,\alpha}$, $N_{x,\alpha}$ define the directions of these uncertainties.

In the following the subscript t is omitted without confusion for typing simplifications.

$$\begin{cases} \dot{x} = (A_\alpha + \Delta_{x,\alpha}N_{x,\alpha})x + B_\alpha u \\ \qquad +E_{d,\alpha}d + E_{f,\alpha}f \\ y = C_\alpha x + D_\alpha u + F_{d,\alpha}d + F_{f,\alpha}f \end{cases} \tag{3}$$

Assumption 1. In this study the pair $(\tilde{A}_\alpha, \tilde{C}_\alpha)$ defined in (9) is assumed observable, or without loss of generality is detectable. It is a standard assumption for all fault detection problems.

That is using Popov criterion, $\forall p$ s.t. $\Re(p) \geq 0$:

$$rank \begin{bmatrix} pI - A_\alpha & -E_f \\ 0 & pI \\ C_\alpha & 0 \end{bmatrix} = n + rank(E_{f,\alpha}) \tag{4}$$

Assumption 2. In this study the system is assumed stable. This assumption is explained with *Theorem 1* and will be used in *Theorem 2*.

Introducing local variable χ, the system can be put in the form:

$$\begin{cases} \dot{x} = A_\alpha x + B_\alpha u + E_{d,\alpha}d + E_{f,\alpha}f + \chi \\ \chi = \Delta_{x,\alpha}N_{x,\alpha}x \\ y = C_\alpha x + D_\alpha u + F_{d,\alpha}d + F_{f,\alpha}f \end{cases} \tag{5}$$

The switched robust PI observer for fault detection and estimation has the form:

$$\begin{cases} \dot{\hat{x}} = A_\alpha\hat{x} + B_\alpha u + L_{P,\alpha}(y - \hat{y}) + E_{f,\alpha}\hat{f} \\ \dot{\hat{f}} = L_{I,\alpha}(y - \hat{y}) \\ \hat{y} = C_\alpha\hat{x} + D_\alpha u \end{cases} \tag{6}$$

where $L_{P,\alpha}$ and $L_{I,\alpha}$ are respectively the proportional and integral gain for the robust PI fault observer.

Assuming the static fault case (i.e. $\dot{f} = 0$), and define the errors $e_x = x - \hat{x}$ and $e_f = f - \hat{f}$, then the following could be written:

$$\begin{aligned} \dot{e}_x &= A_\alpha x + B_\alpha u + E_{d,\alpha}d + E_{f,\alpha}f + \chi \\ &\quad -(A_\alpha\hat{x} + B_\alpha u + L_{P,\alpha}(y - \hat{y}) + E_{f,\alpha}\hat{f}) && \text{(7a)} \\ &= (A_\alpha - L_{P,\alpha}C_\alpha)e_x + E_{f,\alpha}e_f + \chi \\ &\quad +(E_{d,\alpha} - L_{P,\alpha}F_{d,\alpha})d - L_{P,\alpha}F_{f,\alpha}f && \text{(7b)} \\ \dot{e}_f &= -L_{I,\alpha}(y - \hat{y}) \\ &= -L_{I,\alpha}C_\alpha e_x - L_{I,\alpha}F_{d,\alpha}d - L_{I,\alpha}F_{f,\alpha}f && \text{(7c)} \end{aligned}$$

In matrix form:

$$\begin{aligned} \begin{bmatrix} \dot{e}_x \\ \dot{e}_f \end{bmatrix} &= \left(\begin{bmatrix} A_\alpha & E_{f,\alpha} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_{P,\alpha} \\ L_{I,\alpha} \end{bmatrix} [C_\alpha \;\; 0] \right) \begin{bmatrix} e_x \\ e_f \end{bmatrix} \\ &\quad + \begin{bmatrix} E_{d,\alpha} - L_{P,\alpha}F_{d,\alpha} \\ -L_{I,\alpha}F_{d,\alpha} \end{bmatrix} d \\ &\quad + \begin{bmatrix} -L_{P,\alpha}F_{f,\alpha} \\ -L_{I,\alpha}F_{f,\alpha} \end{bmatrix} f + \begin{bmatrix} I \\ 0 \end{bmatrix} \chi && \text{(8a)} \end{aligned}$$

$$r_\alpha = [C_\alpha \;\; 0] \begin{bmatrix} e_x \\ e_f \end{bmatrix} + F_{d,\alpha}d + F_{f,\alpha}f \tag{8b}$$

Let:

$$\begin{aligned} \tilde{A}_\alpha &= \begin{bmatrix} A_\alpha & E_{f,\alpha} \\ 0 & 0 \end{bmatrix}, \tilde{B}_\alpha = \begin{bmatrix} B_\alpha \\ 0 \end{bmatrix}, \tilde{C}_\alpha = [C_\alpha \;\; 0], \\ \tilde{E}_{d,\alpha} &= \begin{bmatrix} E_{d,\alpha} \\ 0 \end{bmatrix}, \tilde{I} = \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ and } L_\alpha = \begin{bmatrix} L_{P,\alpha} \\ L_{I,\alpha} \end{bmatrix} \end{aligned} \tag{9}$$

Then the residual generator in (7) with $\tilde{x} = \begin{bmatrix} e_x^T & e_f^T \end{bmatrix}^T$ becomes:

$$\begin{cases} \dot{\tilde{x}} = (\tilde{A}_\alpha - L_\alpha\tilde{C}_\alpha)\tilde{x} + (\tilde{E}_{d,\alpha} - L_\alpha F_{d,\alpha})d \\ \qquad +(-L_\alpha F_{f,\alpha})f + \tilde{I}\chi \\ r_\alpha = \tilde{C}_\alpha\tilde{x} + F_{d,\alpha}d + F_{f,\alpha}f \end{cases} \tag{10}$$

And the PI observer in (5) with $\bar{x} = \begin{bmatrix} \hat{x}^T & \hat{f}^T \end{bmatrix}^T$:

$$\begin{cases} \dot{\bar{x}} = \tilde{A}_\alpha\bar{x} + \tilde{B}_\alpha u \\ \qquad +L_\alpha F_{d,\alpha}d + L_\alpha F_{f,\alpha}f + L_\alpha\tilde{C}_\alpha\tilde{x} \end{cases} \tag{11}$$

Let: $A_\alpha^* = \tilde{A}_\alpha - L_\alpha\tilde{C}_\alpha$, $E_{d,\alpha}^* = \tilde{E}_{d,\alpha} - L_\alpha F_{d,\alpha}$, $E_{f,\alpha}^* = -L_\alpha F_{d,\alpha}$.

Then the sensitivity functions of disturbance to the residual is defined as:

$$T_{rd_\alpha}(s) = \tilde{C}_\alpha(sI - A_\alpha^*)^{-1}E_{d,\alpha}^* + F_{d,\alpha} \tag{12}$$

The objective of the $\mathcal{H}_\infty$ switched fault dectection PI observer is resumed by the following condition:

$$\|T_{rd_\alpha}\|_\infty < \gamma_\alpha \tag{13}$$

The problem is formulated as following: Find the matrices $L_\alpha = \left[L_{P,\alpha}^T \; L_{I,\alpha}^T\right]^T$, minimizing γ_α such that the switched FD PI observer is stable.

3. PRELIMINARIES

Lemma 1. For any matrices X and Y with appropriate dimensions, the following statement holds:

$$X^T Y + Y^T X < X^T X + Y^T Y \tag{14}$$

Theorem 1. Consider the autonomous system:

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \text{; and the observer } \begin{cases} \dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

And define the error $e_x = x - \hat{x} \Rightarrow \dot{e}_x = (A - LC)e_x$.

Then a suitable Multiple Lyapunov Function (MLF) that ensures the sufficient stability condition of the system and the observer is :

$$V = V_1 + V_2 \tag{15}$$

with $V_1 = \hat{x}^T P \hat{x}$, $V_2 = e_x^T P e_x$, $P > 0$.

Proof 1. The stability condition is $\dot{V} < 0$, then:

$$\begin{gathered} \dot{V} = \dot{V}_1 + \dot{V}_2 < 0, \quad P > 0 \\ (A\hat{x} + LCe_x)^T P\hat{x} + \hat{x}^T P(A\hat{x} + LCe_x) \\ +((A - LC)e_x)^T Pe_x + e_x^T P(A - LC)e_x < 0 \end{gathered}$$

In matrix form:

$$\begin{bmatrix} \hat{x} \\ e_x \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PLC \\ \star & (A - LC)^T P + P(A - LC) \end{bmatrix} \begin{bmatrix} \hat{x} \\ e_x \end{bmatrix} < 0$$

This inequality holds $\forall \left[\hat{x}^T \; e_x^T\right]^T \neq 0$, thus:

$$\begin{bmatrix} A^T P + PA & PLC \\ \star & (A - LC)^T P + P(A - LC) \end{bmatrix} < 0$$

Using Shur complement properties, the diagonal parts should be negative:

$A^T P + PA < 0$, and $(A - LC)^T P + P(A - LC) < 0$ Hence, the observer is stable, and the system is stable as well. $\square$

Theorem 2. For a given uncertain switched system with faults as defined in (3), if there exits a symmetric matrix $P_\alpha > 0$ and positive scalars $\epsilon_{x,\alpha}$ and γ_α, such that the following inequality holds:

$$\begin{bmatrix} \Omega_{d,\alpha} & \Upsilon_{d,\alpha} & P_\alpha \tilde{B}_\alpha & -\tilde{C}_\alpha^T U_\alpha^T & P_\alpha \\ \star & J_{d,\alpha} & 0 & -F_{d,\alpha}^T U_\alpha^T & 0 \\ \star & \star & 0 & \tilde{B}_\alpha^T P_\alpha & 0 \\ \star & \star & \star & \Pi_\alpha & 0 \\ \star & \star & \star & \star & -\dfrac{1}{2}I \end{bmatrix} < 0 \tag{16}$$

where

$$\begin{aligned} \Omega_{d,\alpha} =& P_\alpha \tilde{A}_\alpha + U_\alpha \tilde{C}_\alpha + \tilde{A}_\alpha^T P_\alpha + \\ & \tilde{C}_\alpha^T U_\alpha^T + \tilde{C}_\alpha^T \tilde{C}_\alpha + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \\ \Upsilon_{d,\alpha} =& P_\alpha \tilde{E}_{d,\alpha} + U_\alpha \tilde{F}_{d,\alpha} + \tilde{C}_\alpha^T F_{d,\alpha} \\ J_{d,\alpha} =& F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I, \\ \Pi_\alpha =& \tilde{A}_\alpha^T P_\alpha + P_\alpha \tilde{A}_\alpha + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T + \tilde{C}_\alpha^T \tilde{C}_\alpha \end{aligned}$$

Then a robust PI fault detection observer can be designed where the gain filter $L_\alpha = -P_\alpha^{-1} U_\alpha$

Proof 2. The followings are the constraints for a general robust fault detection observer design:

- If there existe $P_\alpha > 0$, the sufficient stability condition considering the candidate Multiple Lyapunov Function (MLF):

$$V_\alpha = \tilde{x}^T P_\alpha \tilde{x} + \bar{x}^T P_\alpha \bar{x} \tag{17a}$$

$$\dot{V}_\alpha < 0 \tag{17b}$$

- For a positive scalar γ_α, the $\mathcal{H}_\infty$ disturbance rejection condition (13) is formulated as:

$$\left\| r_\alpha|_{f=0} \right\|_2 < \gamma_\alpha \left\| d \right\|_2 \tag{17c}$$

- The boundedness property of the uncertainties is:

$$\begin{aligned} \chi^T \chi =& x^T N_{x,\alpha}^T \Delta_{x,\alpha}^T \Delta_{x,\alpha} N_{x,\alpha} x \\ &< \epsilon^2 x^T N_{x,\alpha}^T N_{x,\alpha} x \end{aligned} \tag{17d}$$

Using $x = e_x + \hat{x} = \bar{I}^T \tilde{x} + \bar{I}^T \bar{x} = \bar{I}^T(\tilde{x} + \bar{x})$, and with the property in *Lemma 1*, we can write:

$$\begin{aligned} x^T N_{x,\alpha}^T N_{x,\alpha} x =& (\tilde{x} + \bar{x})^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T (\tilde{x} + \bar{x}) \\ =& \tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \tilde{x} + \bar{x}^T \bar{I} N_{x,\alpha}^T \bar{I} N_{x,\alpha} \bar{I}^T \bar{x} \\ &+ \tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \bar{x} + \bar{x}^T \bar{I} N_{x,\alpha}^T \bar{I} N_{x,\alpha} \bar{I}^T \tilde{x} \\ <& \tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \tilde{x} + \bar{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \bar{x} \\ &+ \tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \tilde{x} + \bar{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \bar{x} \\ =& 2\tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \tilde{x} + 2\bar{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \bar{x} \end{aligned} \tag{18}$$

Combining the equations (17a) - (17c) yields to:

$$\dot{V}_\alpha + r_\alpha^T r_\alpha - \gamma_\alpha^2 d^T d < 0 \tag{19}$$

Let $V_\alpha = V_{1,\alpha} + V_{2,\alpha}$; $V_{1,\alpha} = \tilde{x}^T P_\alpha \tilde{x}$ and $V_{2,\alpha} = \hat{x}^T P_\alpha \hat{x}$.

Then using the properties (16g) and (17), the general form of the MLF derivatives are:

$$\begin{aligned} \dot{V}_{1,\alpha} =& (A_\alpha^* \tilde{x} + E_{d,\alpha}^* d + E_{f,\alpha}^* f + \bar{I}\chi)^T P_\alpha \tilde{x} \\ &+ \tilde{x}^T P_\alpha (A_\alpha^* \tilde{x} + E_{d,\alpha}^* d + E_{f,\alpha}^* f + \bar{I}\chi) \\ =& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^{*T} P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha (E_{d,\alpha}^* d + E_{f,\alpha}^* f) \\ &+ (E_{d,\alpha} d + E_{f,\alpha}^* f)^T P_\alpha \tilde{x} + \tilde{x}^T P_\alpha \bar{I}\chi + \chi^T \bar{I}^T P_\alpha \tilde{x} \\ <& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha (E_{d,\alpha}^* d + E_{f,\alpha}^* f) \\ &+ (E_{d,\alpha}^* d + E_{f,\alpha}^* f)^T P_\alpha \tilde{x} + \tilde{x}^T P_\alpha P_\alpha^T \tilde{x} + \chi^T \bar{I}^T \bar{I}\chi \\ <& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha (E_{d,\alpha}^* d + E_{f,\alpha}^* f) \\ &+ (E_{d,\alpha}^* d + E_{f,\alpha}^* f)^T P_\alpha \tilde{x} + \tilde{x}^T P_\alpha^2 \tilde{x} + \epsilon_x^2 x^T N_x^T N_x x \\ <& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha)\tilde{x} + \tilde{x}^T P_\alpha (E_{d,\alpha}^* d + E_{f,\alpha}^* f) \\ &+ (E_{d,\alpha}^* d + E_{f,\alpha}^* f)^T P_\alpha \tilde{x} + \tilde{x}^T P_\alpha^2 \tilde{x} \\ &+ \epsilon_x^2 (2\tilde{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \tilde{x} + 2\hat{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \hat{x}) \\ <& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T)\tilde{x} \\ &+ \tilde{x}^T P_\alpha E_{d,\alpha}^* d + d^T E_{d,\alpha}^{*T} P_\alpha \tilde{x} \\ &+ \tilde{x}^T P_\alpha E_{f,\alpha}^* f + f^T E_{f,\alpha}^{*T} P_\alpha \tilde{x} \\ &+ 2\epsilon_x^2 \hat{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \hat{x} \end{aligned} \tag{20}$$

$$
\begin{aligned}
\dot{V}_{2,\alpha} =& \dot{\bar{x}}^T P_\alpha \bar{x} + \bar{x}^T P_\alpha \dot{\bar{x}} \\
=& (\tilde{A}_\alpha \bar{x} + \tilde{B}_\alpha u + L_\alpha \tilde{C}_\alpha \tilde{x} + L_\alpha F_{d,\alpha} d \\
&+ L_\alpha F_{f,\alpha} f)^T P_\alpha \bar{x} + \bar{x}^T P_\alpha (\tilde{A}_\alpha \bar{x} + \tilde{B}_\alpha u \\
&+ L_\alpha \tilde{C}_\alpha \tilde{x} + L_\alpha F_{d,\alpha} d + L_\alpha F_{f,\alpha} f) \\
=& \bar{x}^T (\tilde{A}_\alpha^T P_\alpha + P_\alpha \tilde{A}_\alpha) \bar{x} + \bar{x}^T P_\alpha \tilde{B}_\alpha u + u^T \tilde{B}_\alpha^T P_\alpha \bar{x} \\
&+ \bar{x}^T P_\alpha L_\alpha \tilde{C}_\alpha \tilde{x} + \tilde{x}^T \tilde{C}_\alpha^T L_\alpha^T P_\alpha \bar{x} + \bar{x}^T P_\alpha L_\alpha F_{d,\alpha} d \\
&+ d^T F_{d,\alpha}^T L_\alpha^T P_\alpha \bar{x} + \bar{x}^T P_\alpha L_\alpha F_{f,\alpha} f \\
&+ f^T F_{f,\alpha}^T L_\alpha^T P_\alpha \bar{x}
\end{aligned}
\tag{21}
$$

In the fault free case, the inequalities (19)-(21) yield to:

$$
\begin{aligned}
&\dot{V}_\alpha|_{f=0} + r_\alpha^T r_\alpha - \gamma_\alpha^2 d^T d \\
<& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T) \tilde{x} \\
&+ \tilde{x}^T P_\alpha E_{d,\alpha}^* d + d^T E_{d,\alpha}^{*T} P_\alpha \tilde{x} + 2\epsilon_x^2 \bar{x}^T \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \bar{x} \\
&+ \bar{x}^T (\tilde{A}_\alpha^T P_\alpha + P_\alpha \tilde{A}_\alpha) \bar{x} + \bar{x}^T P_\alpha \tilde{B}_\alpha u \\
&+ u^T \tilde{B}_\alpha^T P_\alpha \bar{x} + \bar{x}^T P_\alpha L_\alpha \tilde{C}_\alpha \tilde{x} + \tilde{x}^T \tilde{C}_\alpha^T L_\alpha^T P_\alpha \bar{x} \\
&+ \tilde{x}^T \tilde{C}_\alpha^T F_{d,\alpha} d + d^T F_{d,\alpha}^T \tilde{C}_\alpha \tilde{x} + d^T F_{d,\alpha}^T F_{d,\alpha} d \\
&+ \bar{x}^T P_\alpha L_\alpha F_{d,\alpha} d + d^T F_{d,\alpha}^T L_\alpha^T P_\alpha \bar{x} \\
&+ \tilde{x}^T \tilde{C}_\alpha^T \tilde{C}_\alpha \tilde{x} - \gamma_\alpha^2 d^T d \\
<& \tilde{x}^T (P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T + \tilde{C}_\alpha^T \tilde{C}_\alpha) \tilde{x} \\
&+ \tilde{x}^T (P_\alpha E_{d,\alpha}^* + \tilde{C}_\alpha^T F_{d,\alpha}) d + d^T (E_{d,\alpha}^{*T} P_\alpha + F_{d,\alpha}^T \tilde{C}_\alpha) \tilde{x} \\
&+ \bar{x}^T (\tilde{A}_\alpha^T P_\alpha + P_\alpha \tilde{A}_\alpha + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T) \bar{x} + \bar{x}^T P_\alpha \tilde{B}_\alpha u \\
&+ u^T \tilde{B}_\alpha^T P_\alpha \bar{x} + \bar{x}^T P_\alpha L_\alpha \tilde{C}_\alpha \tilde{x} + \tilde{x}^T \tilde{C}_\alpha^T L_\alpha^T P_\alpha \bar{x} \\
&+ \bar{x}^T P_\alpha L_\alpha F_{d,\alpha} d + d^T F_{d,\alpha}^T L_\alpha^T P_\alpha \bar{x} \\
&+ d^T (F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I) d = \Gamma_\alpha|_{f=0} < 0
\end{aligned}
\tag{22}
$$

Solving the set of inequalities $\Gamma_\alpha|_{f=0} < 0$ guarantees the solution for (19).

In the quadratic form:

$$
\begin{bmatrix} \tilde{x} \\ d \\ u \\ \bar{x} \end{bmatrix}^T
\begin{bmatrix}
\Omega_{d,\alpha}^* & \Upsilon_{d,\alpha}^* & P_\alpha \tilde{B}_\alpha & \tilde{C}_\alpha^T L_\alpha^T P_\alpha \\
\star & J_{d,\alpha} & 0 & F_{d,\alpha}^T L_\alpha^T P_\alpha \\
\star & \star & 0 & B_\alpha^T P_\alpha \\
\star & \star & \star & \Pi_\alpha
\end{bmatrix}
\begin{bmatrix} \tilde{x} \\ d \\ u \\ \bar{x} \end{bmatrix} < 0
\tag{23}
$$

where

$$
\begin{aligned}
\Omega_{d,\alpha}^* &= P_\alpha A_\alpha^* + A_\alpha^* P_\alpha + 2P_\alpha^2 + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T + \tilde{C}_\alpha^T \tilde{C}_\alpha, \\
\Upsilon_{d,\alpha}^* &= P_\alpha E_{d,\alpha}^* + \tilde{C}_\alpha^T F_{d,\alpha}, \\
J_{d,\alpha} &= F_{d,\alpha}^T F_{d,\alpha} - \gamma_\alpha^2 I, \\
\Pi_\alpha &= \tilde{A}_\alpha^T P_\alpha + P_\alpha \tilde{A}_\alpha + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T + \tilde{C}_\alpha^T \tilde{C}_\alpha
\end{aligned}
$$

This inequality holds $\forall \begin{bmatrix} \tilde{x}^T & d^T & u^T & \bar{x}^T \end{bmatrix}^T \neq 0$, thus:

$$
\begin{bmatrix}
\Omega_{d,\alpha}^* & \Upsilon_{d,\alpha}^* & P_\alpha \tilde{B}_\alpha & \tilde{C}_\alpha^T L_\alpha^T P_\alpha \\
\star & J_{d,\alpha} & 0 & F_{d,\alpha}^T L_\alpha^T P_\alpha \\
\star & \star & 0 & \tilde{B}_\alpha^T P_\alpha \\
\star & \star & \star & \Pi_\alpha
\end{bmatrix} < 0
\tag{24}
$$

This BMI is transformed into LMI by replacing $U_\alpha = -P_\alpha L_\alpha$, and using Schur complement formula for $P_\alpha^T P_\alpha$. It follows:

$$
\begin{bmatrix}
\Omega_{d,\alpha} & \Upsilon_{d,\alpha} & P_\alpha \tilde{B}_\alpha & -\tilde{C}_\alpha^T U_\alpha^T & P_\alpha \\
\star & J_{d,\alpha} & 0 & -F_{d,\alpha}^T U_\alpha^T & 0 \\
\star & \star & 0 & \tilde{B}_\alpha^T P_\alpha & 0 \\
\star & \star & \star & \Pi_\alpha & 0 \\
\star & \star & \star & \star & -\frac{1}{2} I
\end{bmatrix} < 0
\tag{25}
$$

$$
\begin{aligned}
\Omega_{d,\alpha} =& P_\alpha \tilde{A}_\alpha + U_\alpha \tilde{C}_\alpha + \tilde{A}_\alpha^T P_\alpha + \\
& \tilde{C}_\alpha^T U_\alpha^T + \tilde{C}_\alpha^T \tilde{C}_\alpha + 2\epsilon_x^2 \bar{I} N_{x,\alpha}^T N_{x,\alpha} \bar{I}^T \\
\Upsilon_{d,\alpha} =& P_\alpha \tilde{E}_{d,\alpha} + U_\alpha F_{d,\alpha} + \tilde{C}_\alpha^T F_{d,\alpha}
\end{aligned}
$$

□

Remark 1. In this theorem, a MLF has been used. The second term V_2 has been added to ensure the feasibility of the LMI (16). Doing that adds the *Assumption 2* on the stability of the system. We can consider that our interest here focuses on stable systems, and we work only on the observability and FD problems.

Theorem 3. For a given square $n \times n$ matrix A_α, if there exits a symmetric matrix $P_\alpha > 0$ and a positive scalar ξ_α such that the following inequality holds:

$$
A_\alpha^T P_\alpha + P_\alpha A_\alpha - 2\xi_\alpha P_\alpha < 0 \tag{26}
$$

Then all eigenvalues of A_α are on left plane of ξ_α.

Proof 3. (26) is a result of a classical Lyapunov function for sufficient condition of stability.

The system $\dot{x} = (A_\alpha - \xi_\alpha I)x$ is stable if there exist a symmetric matrix $P_\alpha > 0$ where $V = x^T P_\alpha x$, $\dot{V} < 0$.

Thus:

$$
(A_\alpha - \xi_\alpha I)^T P_\alpha + P_\alpha (A_\alpha - \xi_\alpha I) < 0 \tag{27}
$$

which is equivalent to (26).

□

Remark 2. The rise time propertie of the system is the time that takes to reach 90% of the steady state, it is approximated in the first order system to:

$$
t_{rise,\alpha} = \frac{2.2}{\xi_\alpha} \tag{28}
$$

This shows the strong connection between the dominant pole region (ξ_α) and the time specification.

4. PI ROBUST FAULT DETECTION OBSERVER DESIGN

The multi-objectives of the observer are: (a) robustness against uncertainties, (b) robustness and perturbation rejection, (c) sensitivity toward faults and (d) a correct time response for fault detection. In order to meet all these constraint, the developed design in this section can be adopted.

Whilst the raise time constant is predefined and is inversely proportional to ξ_α, the coefficient γ_α has to be minimized. This consists in solving a set of LMIs as an optimization problem.

These LMIs are :

$$
P_\alpha \tilde{A}_\alpha + \tilde{A}_\alpha^T P_\alpha + U_\alpha \tilde{C}_\alpha + \tilde{C}_\alpha^T U_\alpha^T - 2\xi_\alpha P_\alpha < 0 \tag{29}
$$

$$
P_\alpha > 0 \tag{30}
$$

and the same LMI in (24).

The gain filter is $\bar{L}_\alpha = -P_\alpha^{-1} U_\alpha$.

Using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs can then be solved minimizing γ_α^2.

Symbol	Variable	Value	Unit
v	longitudinal velocity	-	[m/s]
u_L	steering angle	-	[rad]
β	side slip angle	-	[rad]
$\dot{\psi}$	yaw rate	-	[rad/s]
γ_L	lateral acceleration	-	[m/s^2]
m	Mass of the vehicle (+2 passengers)	1417	[kg]
I_z	yaw moment of inertia	1808	[kg.m]
c_f	front corners stiffness	41654	[N/rad]
c_r	rear corners stiffness	56862	[N/rad]
l_f	distance form CoG to front axle	1.12	[m]
l_r	distance form CoG to rear axle	1.46	[m]

Table I. Notations and vehicle parameters

The objectives (a) and (b) are guaranteed by LMIs (**??**) and (30). The objective (c) is resolved by the integral term of the PI observer: when stable, it will converges to the correct value of the fault. The objective (d) is guaranteed as well by LMIs (29) and (30), it derives from *Theorem 3*, where BMIs are transformed into LMI as in final steps of the *Proof 1*.

Finally, the designed observer can be put in the following form:

$$\left\{ \begin{aligned} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{f}} \end{bmatrix} &= \begin{bmatrix} A_\alpha - L_{P,\alpha}C_\alpha & E_{f,\alpha} \\ -L_{I,\alpha}C_\alpha & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{f} \end{bmatrix} \\ &+ \begin{bmatrix} B_\alpha - L_{P,\alpha}D_\alpha & L_{P,\alpha} \\ -L_{I,\alpha}D_\alpha & L_{I,\alpha} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \end{aligned} \right. \tag{31}$$

5. EXAMPLE

5.1 Bicycle switched model

Consider the problem of the FD in the lateral control of a vehicle.

The bicycle-model is widely used as representation of the system Mammar and Koenig (2002). Though, this model is non-linear since it has $\frac{1}{v}$ and $\frac{1}{v^2}$ terms in it:

$$\begin{bmatrix} \dot{\beta}(t) \\ \ddot{\psi}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{c_r + c_f}{mv(t)} & \dfrac{c_r l_r - c_f l_f}{mv^2(t)} - 1 \\ \dfrac{c_r l_r - c_f l_f}{I_z} & -\dfrac{c_r l_r^2 + c_f l_f^2}{I_z v(t)} \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \begin{bmatrix} \dfrac{c_f}{mv} \\ \dfrac{c_r l_f}{I_z} \end{bmatrix} u_L(t) + \begin{bmatrix} \dfrac{1}{mv} \\ \dfrac{l_w}{I_z} \end{bmatrix} F_w(t) \tag{32}$$

$$y = \begin{bmatrix} \dfrac{c_r + c_f}{m} & \dfrac{c_f l_f - c_r l_r}{m} \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \frac{c_f}{m} u_L(t) \tag{33}$$

The measured output is the lateral acceleration γ_L, the entry command is the steering angle u_L, the sates are the side slip angle β and the yaw rate $\dot{\psi}$, and we consider the wind force as a perturbation signal F_w.

In this approach, the system is linearized around multiple points v_α as shown in dashed curve of figure 3. It is calculated as the integer part of the output of the division: $\frac{v(t)}{\delta}$. v_α is the switching signal.

Using a Taylor expansion in the neighborhood of v_α:

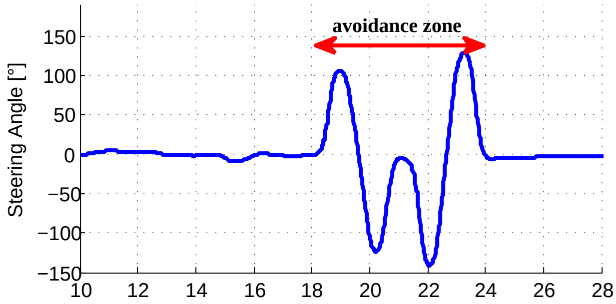

Fig. 1. Steering Angle [°]

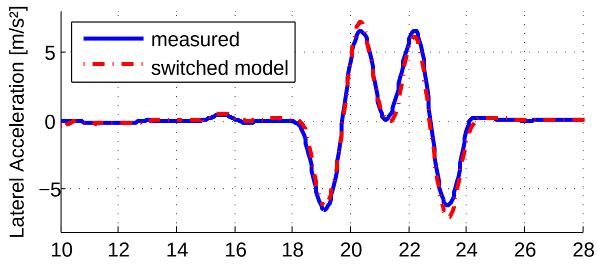

Fig. 2. Lateral acceleration [m/s^2]

$$\frac{1}{v}|_{v=v_\alpha} = \frac{1}{v_\alpha} - \frac{1}{v_\alpha^2}(v - v_\alpha) + \mathcal{O}(\frac{1}{v^2}) \tag{34}$$

$$\frac{1}{v^2}|_{v=v_\alpha} = \frac{1}{v_\alpha^2} - \frac{2}{v_\alpha^3}(v - v_\alpha) + \mathcal{O}(\frac{1}{v^3}) \tag{35}$$

Then

$$A = \underbrace{A_0 + \frac{1}{v_\alpha}A_1 + \frac{1}{v_\alpha^2}A_2}_{A_\alpha} + \underbrace{(-\frac{1}{v_\alpha^2}A_1 - \frac{2}{v_\alpha^3}A_2)}_{N_{x,\alpha}} \underbrace{(v - v_\alpha)}_{\Delta_{x,\alpha}} \tag{36}$$

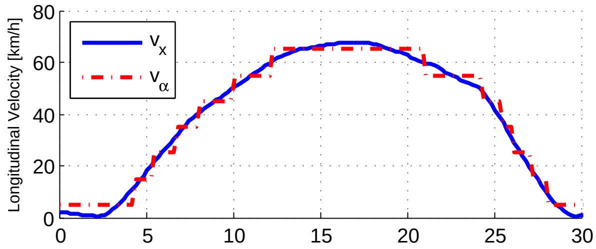

Fig. 3. Longitudinal velocity [km/h] and switching signal

5.2 Simulation scenario

Experimental data have been taken from a "Renaul Scenic" car, provided by the french laboratory MIPS. In this example, we consider the scenario of the evasive manoeuvre test more commonly known as the moose test.

Figure 1 shows the measured steering angle for this scenario. The avoidance occurs between $t = 18$ to $t = 24s$.

On figure 2, a comparison between the measured data and the simulated switched bicycle model for lateral acceleration validates the switched model, it can be used

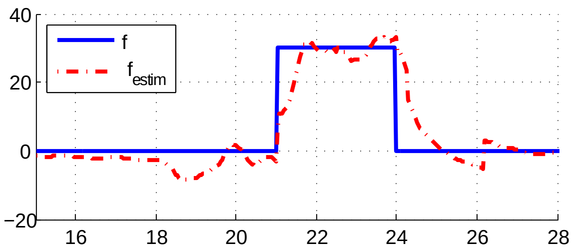

Fig. 4. Fault and estimated fault signals

then for the observer design. Both the longitudinal velocity (continuous blue) and the operating points (dashed red) are plot on figure 3. As one can observe, $\delta = 10$ km/h. As these tests were done up to 100 km/h, 10 observers are calculated. The uncertainties of ± 5 km/h guarantee the continuity of the switched observers.

5.3 Simulation Results

The fault considered in this application is an actuator fault, that occurs on the actuator. The uncertain switched state space representation in this case becomes :

$$\begin{cases} \dot{x} = (A_\alpha + \Delta_{x,\alpha} N_{x,\alpha})x \\ \qquad +B_\alpha(u+f) + E_{d,\alpha} d \\ y = Cx + D(u+f) \end{cases} \tag{37}$$

Applying the set of LMIs detailed in section 4, a robust proportional integral fault observer can be designed, meeting the desired objectives.

For a fault that occurs between $t = 21$ and $t = 24$s, we can see on figure 4 the fault estimation by the switched observer. The estimated singal rises within 1s.

In the non-faulty case, the estimated signal is very low, but non-zero. This is due to the uncertainties, but also to the parameters variation and the sensors calibration in the vehicle. These parameters can never be accurately known, the design of the oberver is robust to these though.

6. CONCLUSION AND FURTHER WORK

The technique presented in this paper provides a framework for generating a class of robust fault detection observers.

Indeed, we have analyzed the problem of fault detection using proportional integral observer. We showed that the H_∞ PI observer structure formulated using LMIs makes it possible to decouple the disturbances while estimating the states and faults with satisfactory convergence properties.

Several time- and frequency-domain specifications have been expressed as LMI constraints on the observers state-space matrices. These analysis are then used for multi-objective synthesis purposes. A compromise of these objectives is proposed as a criterion to minimize. A LMIs feasibility problem is formulated. It is then solved as optimization problem by using efficient LMI solver. A partical example with field data was given to illustrate and validate this approach.

In future work, this design can be extended to a class of non-linear Lipschitz sytem, were the same idea of boundedness can be used.

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